1. PHD PROJECT DESCRIPTION (4000 characters max., including the aims and work plan)

Project title:
Singularity formation in the vortex patch dynamics of the 2D Euler equation

1.1. Project goals

1. Find finite-time singularities developed by solutions of a semiflow approximating contour dynamics of the 2D Euler equation. In particular, confirm that polygonal lines in the complex plane are examples of such singularities.
2. Find the asymptotic expansions of the solutions developing the finite-time sigularities from the previous point.

1.2. Outline

We are concerned with the geometric flow

$$z_t=-z_{xxx}+(z^*)_x\,(z_{xx})^2, \qquad (1)$$

where z(t,x) is a time-dependent family of regular curves contained in the complex plane. The flow was derived in [1] as a formal approximation of a non-local differential equation, that describes the evolution of patches of constant vorticity for the incompressible 2D Euler equation. In this project, we intend to look for the finite-time singularities developed by the solutions of the geometric flow (1). An example of such singularities are *double spirals* represented by the following curve

$$z_0(x)=x(1+\mu^2)^{-1/2}\exp(i(\theta_{+/-}-\mu\ln|x|)) \quad \text{for} \quad \pm x>0, \quad (2)$$

where $\mu \in R$ and $\theta_+\ \theta_- \in [0,2\pi)$ are such that $|\theta_+ - \theta_-| \neq \pi$. The singularity $z_0(x)$ consists of two congruent by rotation logarithmic spirals and frequently appears in the motion of a turbulent flow. In [2], the perturbation argument and ODEs techniques were used to show that, the double spiral is a finite time singularity of the flow (1) if the value $|\theta_+ - \theta_-| + |\mu|$ is sufficiently close to zero. The restriction on the quantity, were removed in [3], where it was shown that arbitrary double spiral is developed by a self-similar solution, whose profile function is an appropriate Ablowitz-Segur solution of the PII equation. In particular, for $\mu = 0$, the singularity (2) reduces to a corner consisting of two rays starting from the origin. The first aim of the project is to find subsequent singularities developed by a solution of the flow (1). In particular, we expect to show that polygonal lines in the complex plane are their particular cases. Obtained singularities will be considered more carefully in the second gole of the project, where we are going to derive asymptotic expansions of the corresponding solutions.

1.3. Work plan

1. Perform analysis of the geometric flow and the mKdV equaiton to understand structure of their solutions
2. Use the Riemann-Hilbert approach to indicate Painlevé II transcendents such that their linear combination, up to some remainder term, is the solutions of the geometric flow (1).
3. Find appropriate norms and use them to obtain estimates for the remainder term obtained in the previous point.

1.4. Literature

1. R.E. Goldstein, D.M. Petrich, Solitons, Euler's equation, and vortex patch dynamics, Phys. Rev. Lett. 69 (1992), no. 4, 555-558.

2. G. Perelman, L. Vega, Self-similar planar curves related to modified Korteweg-de Vries equation, J. Differential Equations 235 (2007), no. 1, 56-73.
3. K. Dunst, P. Kokocki, Double spiral singularities for a flow related with the 2D Euler equation – submitted
4. S. Gutierrez, L. Vega, Self-similar solutions of the localized induction approximation: singularity formation, Nonlinearity 17, 2091–2136.
5. V. Banica, L. Vega, Evolution of polygonal lines by the binormal flow, arXiv:1807.06948

1.5. Required initial knowledge and skills of the PhD candidate

- Analytical thinking
- Willingness to self-study
- Understanding of mathematical analysis
- Basic knowledge of general topology, functional analysis, complex analysis and partial differential equations

1.6. Expected development of the PhD candidate's knowledge and skills

The candidate has a wide knowledge in the following fileds of mathematics

- Well-posedness for the modified Korteweg-de Vries equation in the classical Sobolev spaces
- Dynamics of vortex patches for the 2D Euler equation
- Riemann-Hilbert approach to the Painlevé II equation
- Singular integral operators defined on the contours in the complex plane (in particular Cauchy operator and Hilbert transform

Furthermore the candidate understands

- Steepest-descent method
- Inverse monodromy method