1. PHD PROJECT DESCRIPTION

Project title: ***Równania różniczkowe cząstkowe z operatorami nielokalnymi: podejście analityczne i probabilistyczne.***

1.1. Project goals

For the last two decades, the interest in partial differential equations (PDEs) with non-local (integro-differential) operators has grown rapidly (see Literature). The recent surge in interest in this topic has been caused by the growing number of scientific publications, revealing that in a large part of physical, biological, chemical and mathematical finance models, the substitution of classical differential operators by non-local operators in related PDEs leads to the solutions that better describe the phenomena, both locally and globally.

The basic difference between local and non-local operator is that the evaluation of non-local operator on a function u at point x depends on values of u on the whole space and not, as in the case of local operator, on values of u on an arbitrary small neighborhood of x. The model example of a non-local operator is the fractional Laplacian which is a fractional power of classical Laplacian. In the project, we would like to use an approach to the above-mentioned PDEs, which is a mix of analytical and probabilistic tools. It is based on one of the most intriguing correspondence in mathematics: between a class of integro-differential operators and a class of Markov stochastic processes (Feynman-Kac formula). We would like to study PDEs with two classes of operators. The first one, we call it (A), is the class of Levy-type operators (non-divergence form elliptic operators), and the second one, we call it (B), consists of semi-Dirichlet operators (divergence form elliptic operators). The purpose of the present project is twofold. Firstly, we aim to solve, using probabilistic and analytical tools, some open problems in the theory of PDEs with integro-differential operators of the class (A), (B). Secondly, we want to further explore the relation between PDEs and stochastic analysis to search for effective links that may lead to new interesting methods and results, both in PDEs and stochastic analysis.

1.2. Outline

We would like to focus on the following tasks involving operator A from the classes (A), (B):

1) ***Strong maximum principle for nonlinear Schrodinger equations***.
In 1984 Vazquez published an impactful paper [29] on strong maximum principle for positive supersolutions to $-\Delta w + \beta(w) = 0$, in D. Here, $\beta : [0,\infty) \to \mathbb{R}$, $\beta(0) = 0$, and β is non-decreasing. The goal of the project is to give a necessary and sufficient condition for SMP to hold for positive supersolutions to $-Aw + \beta(w) = 0$, in D;

2) ***The existence of solutions to Schrodinger equations with singular potential.***
Subsequently, we investigate the existence problem for singular Schrodinger equations of the form $-Au + u \cdot \nu = \mu$, where ν is a positive Borel measure absolutely continuous with respect to a capacity, and μ is a bounded Borel measure. This type of equations have numerous applications in variety of models in nuclear physics, solid-state physics, and quantum field theory (see [3]);

3) ***Kato's inequality.***
Assume that there exists the Green function for $-A$. Prove Kato's inequality and Grishin's lemma in case Au is a Radon measure. The second problem which we would like to investigate within Task 3 is Kato's inequality up to the boundary. For the first time this issue was considered by Brezis and Ponce in [9] and next improved by Ancona in [2];

4) ***Removable and isolated singularities.***
We would like to focus on the following problem considered by P. Lions in [20] and later studied in [4,10,15,18]. Suppose that u is a positive classical solution to $-Au = u^p$ *in* $D \backslash K$. Here $p > 1$ and K is a closed set of capacity zero. The problem is to determine the behavior of u on the whole D. The goal of Task 4 is to provide a unified method of dealing with the mentioned problem under assumption of the existence of the Green function for $-A$*;*

5) ***Eigenfunctions for nonlinear Schrodinger equations.***
One of the fundamental parts of the theory of PDEs is the eigenvalue problem. By Jentzsch's theorem: if Markov semigroup (T_t) generated by A is a compact contraction on L^p for some $p \geq 1$, and it is irreducible, then there exists a unique strictly positive eigenfunction for $-A$ (so called principle eigenfunction).
The problem is more complex when we perturb $-A$ by a function V (not necessarily positive, see [8]). Assume that there exists a strictly positive principle eigenvalue for $-A$. Problem: formulate conditions on V under which there exists a strictly positive eigenfunction for $-A + V$. The next goal within Task 5 is to study the existence of strictly positive solutions to the following equation $-Au + bf(\cdot, u) = mu,$ where $b \geq 0, m$ are bounded measurable functions and f has superlinear growth (see [17]);

6) ***The Hopf lemma.***
The aim is to formulate and prove Hopf's lemma for subsolutions of integro-differential PDEs with operators from the class (A), (B). In the case of non-local operators, however, we can not expect that the similar to the classical one form of Hopf's lemma will hold: $u_\alpha(x) = -c_\alpha(1 - |x|^2)^\alpha$ is a classical (not even Lipschitz up to the boundary) solution to $-\Delta^\alpha u_\alpha = -1$ on the unit ball (smooth domain). Therefore, to describe the behavior of subsolutions near the boundary of D in the case of non-local operators, we need to use

some other formulation of Hopf's lemma which does not involve classical derivative of functions.

1.3. Work plan

See Sections 1.1 and 1.2.

1.4. Literature

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1.5. **Required initial knowledge and skills of the PhD candidate:** analytic thinking, willingness of self-study, understanding of mathematical analysis, solid knowledge on differential equations, functional analysis, topology, and basic knowledge on probability theory.

1.6. **Expected development of the PhD candidate's knowledge and skills:** conducting research on a high level, acquiring skills to present scientific achievements on professional level, acquiring advanced knowledge on potential theory, stochastic equations, integro-differential operators, non-local PDEs.