1. **PHD PROJECT DESCRIPTION (4000 characters max., including the aims and work plan, all in English)**

Project title: Ergodic theory, B-free systems and multiplicative number theory

1.1. Project goals

A) Study classes of automorphisms disjoint from all ergodic automorphisms/ disjoint from all totally ergodic automorphisms. Questions of particular interest are: entropy questions (zero entropy is expected), spectral questions (what kind of spectral types can we achieve in these classes), (countr)examples and constructions, stability on extensions (both distal and relatively weakly mixing), solution of the problem of multipliers.

B) Study dynamical properties of B-free systems. Problems and questions of particular interest are: (notion and) characterization of tautness for higher dimensional analogues of B-free systems, in particular, in the context of number fields; search for a multi-dimensional counterpart of Davenport-Erdos Theorem; study of the minimal case in the context of entropy and invariant measures (in particular to which extent minimality implies unique ergodicity, study admissible subshifts); characterization of Toeplitz sequences which appear as B-free systems.

1.2. Outline

Our starting point and a motivation for the doctoral project is Sarnak's conjecture [10] from 2010 on which we now detail. Namely, the set P={2,3,5,...} of prime numbers is commonly believed to have all attributes of a random subset of the set N of natural numbers. If so, also arithmetic functions (that is, sequences of complex numbers) which somehow ``determine" the primes should also behave like random functions (random variables using the language of probability). As each natural number factors uniquely into primes we think about arithmetic functions determined by their values on the primes (and their powers). This leads to the concept of a multiplicative function u:N->C for which u(mn)=u(m)u(n) whenever m,n are coprime. Not surprisingly, many classical arithmetic functions are multiplicative, prominent examples of such are the Moebius function \mu and the Liouville function \lambda. If we persist in the probabilistic way of thinking, and we want to view \mu as a random function then (motivated by the classical central limit theorem) we should see a square root type of cancellations of 1s and -1s. But this is already equivalent to the Riemann Hypothesis. Another approach which is the key Sarnak's idea is to look at (bounded) multiplicative function from the point of view of dynamics, see [2]. So, we look at \mu as a point in the subshift determined by it and we ask what kind of (invariant) measures it (quasi)-generates. This leads to ergodic theory questions and Sarnak's conjecture predicts that randomness of \mu yields that all such measure-theoretic dynamical systems are close to be disjoint (in the Furstenberg sense) with all deterministic (zero entropy) systems. As formulated, Sarnak's conjecture looks weaker that the ``natural" claim (if we believe in the ``randomness law") that \mu is generic for precisely one measure and the corresponding measure-theoretic system is relatively Bernoulli over the factor determined by |\mu|. This strong statement known under the name

of Chowla conjecture was formulated in 1965. In fact, due to works of Tao [11] and others [5], we know by now that the two conjectures are almost equivalent. As noticed by Sarnak, there is a bridge between this and the theory of B-free systems: |\mob| determines so called square-free system which is a topological factor of the Moebius subshift. This results in many related questions on other arithmetic functions coming from the theory of B-free systems. The doctoral project goes now in two directions: ergodic theory and the theory of B-free systems. As for ergodic theory aspects, following the work of Frantzikinakis and Host [4] at least when we study the logarithmic version of Chowla conjecture, this leads to the theory of dynamical systems generated by so called stationary processes [6]. More precisely, for a class of multiplicative functions the corresponding Furstenberg systems are either determined by strongly stationary processes or stationary processes which are strongly stationary along an arithmetic sequence. Moreover, due to a theorem of Frantzikinakis [3], we may think of Fursenberg systems as either ergodic, then the Chowla conjecture holds, or non-ergodic, but the latter implies that the ergodic decomposition is extremely non-trivial. All the above leads to a purely ergodic aim to understandtwo classes of dynamical systems: Erg^\perp and TotErg^\perp, i.e. systems that are disjoint with all ergodic systems or disjoint from all totally ergodic systems, respectively. These classes were not studied before and we list more particular problems in ``Project goals". Let us turn now to B-free systems. While B-free sets were studied in number theory for almost 100 years it is only recently they attract a lot of attention from the dynamical point of view, see [2] for a recent development. In particular, Keller [7] and Kułaga-Przymus with Lemańczyk (jr.) [10] proved an elegant dynamical characterization of tautness (which is a purely arithmetic property): the set of B-free numbers is taut if and only if the Mirsky measure (naturally generated by the characteristic function of the set of B-free numbers) has full support. This amazing characterization is naturally expected to have an extension of higher dimensional systems, i.e. for lattices Or in the context of number fields. This direction of study also includes a search for a multi-dimensional counterpart of Davenport-Erdos theorem. One of the tools should be a new 0-1 law of Kolmogorov-Keller. Among other problems we mention the zero entropy problem in the minimal case. It is well-known by now by the results of Toruń school [1] and Keller that minimality is equivalent to the characteristic function of the set of B-free numbers being Toeplitz. However, what kind of Toeplitz sequences can be realized in this way is completely open. Finally, despite the fact that many things are known about invariant measures for the hereditary closures of B-free systems [1], [8], it is still not understood which of them are supported by the B-free systems themselves. For example, we do not know to which extent minimality implies unique ergodicity for such systems. It seems also to be interesting to study so called admissible subshifts which contain B-free subshifts and are close to a basic concept of admissibility in number theory.

1.3. Work plan

(i) Study the class of automorphisms disjoint from all ergodic automorphisms (entropy, distal and relatively weakly mixing construction, multiplier problem).

(ii) Study the class of automorphisms disjoint from all totally ergodic automorphisms.

(iii) Find analogues and prove a relevant characterization of tautness for B-free systems in number fields.

(iv) Study dynamical properties of minimal B-free systems (entropy, invariant measures, Toeplitz property).

(v) Study dynamical properties of admissible subshifts.

1.4. Literature

[1] A. Dymek, S. Kasjan, J. Kułaga-Przymus, M. Lemańczyk, B-free sets and dynamics, Trans. Amer. Math. Soc. 370 (2018), 5425-5489.

[2] S. Ferenczi, J. Kułaga-Przymus, M. Lemańczyk, Sarnak's Conjecture -- what's new, in:Ergodic Theory and Dynamical Systems in their Interactions with Arithmetics and Combinatorics, CIRM Jean-Morlet Chair, Fall 2016, Editors: S. Ferenczi, J. Kułaga-Przymus, M. Lema\'nczyk, Lecture Notes in Mathematics 2213, Springer International Publishing, pp. 418

[3] N. Frantzikinakis, Ergodicity of the Liouville system implies the Chowla conjecture, Discrete Anal. 2017, Paper No. 19, 41 pp.

[4] N. Frantzikinakis, B. Host, The logarithmic Sarnak conjecture for ergodic weights, Ann. of Math. (2) 187 (2018), no. 3, 869–931.

[5] A. Gomilko, D. Kwietniak, M. Lemańczyk, Sarnak's conjecture implies the Chowla conjecture along a subsequence}, in: Ergodic theory and dynamical systems in their interactions with arithmetics and combinatorics, 237–247, Lecture Notes in Math., 2213, Springer, Cham, 2018.

[6] E. Jenvey, Strong stationarity and de Finetti's theorem, J. Anal. Math. 73 (1997), 1–18.

[7] G. Keller, Generalized heredity for B-free sets, Studia Math. 2020.

[8] J. Kułaga-Przymus, M. Lemańczyk, B. Weiss, Invariant measures for B-free systems, Proc. Lond. Math. Soc. (3) 110 (2015), 1435–1474.

[9] J. Kułaga-Przymus, M.D. Lemańczyk, Hereditary subshifts whose measure of maximal entropy has no Gibbs property, arXiv 2020.

[9] P. Sarnak, Three lectures on the Moebius function, randomness and dynamics http://publications.ias.edu/sarnak/.

[10] T. Tao, Equivalence of the logarithmically averaged Chowla and Sarnak conjectures, Number Theory – Diophantine Problems, Uniform Distribution and Applications: Festschrift in Honour of Robert F. Tichy's 60th Birthday (C. Elsholtz and P. Grabner, eds.), Springer International Publishing, Cham, 2017, pp. 391–421.

1.5. Required initial knowledge and skills of the PhD candidate: analytical thinking, willingness to self-study, understanding of mathematical analysis, basic knowledge topology, spectral theory, dynamics and number theory.

1.6. Expected development of the PhD candidate's knowledge and skills: conducting research on a high level, acquiring skills to present scientific achievements on professional level, acquiring advanced knowledge in dynamics, spectral theory and number theory.