

## 1. PHD PROJECT DESCRIPTION (4000 characters max., including the aims and work plan)

**Project title:** *Backward stochastic differential equations and their applications to optimization problems*

### 1.1. Project goals

The goal of the project is to provide a series of existence and uniqueness results for Backward Stochastic Differential Equations (BSDEs), Reflected BSDEs (RBSDEs), second order BSDEs and second order RBSDEs, and then to show the relations of solutions of these problems with value functions in various optimization problems (see Outline) which, in a natural manner, arises in mathematical finance, economy and engineering. Some of these relations are already known and we continue to study on them, whereas some are poorly understood and create challenging problems in the project. As to the optimization problems, we are interested in the existence, regularity, stability and approximation of the value functions, optimal and  $\varepsilon$ -optimal strategies. We also pay special attention to Markov models and connections of the value functions with Hamilton-Jacobi-Bellman partial differential equations (viscosity and mild solutions).

### 1.2. Outline

In 1990, Pardoux and Peng introduced in [25] the notion of nonlinear BSDEs on a probability space with filtration generated by a given Wiener process. Independently, in 1992 Duffie and Epstein [8, 9] presented a BSDEs formulation of recursive utility. In 1995-1997, El Karoui et al. [11] and Cvitanic and Karatzas [5] generalized the notion of BSDEs and introduced reflected BSDEs. In both papers, the authors provided formulas that link RBSDEs with optimization problems. Nowadays, the theory of (Reflected) BSDEs and their numerous variants are recognized as very powerful tools in many branches of mathematics, especially in optimization problems and PDEs (see Literature).

The aim of the project is to develop the theory of BSDEs and RBSDEs towards applications in some crucial optimization problems. We will study the following topics:

- 1) BSDEs, second order BSDEs (see [31]) on general filtered spaces and control problems  
- the player optimizes its value function (reward) according to the parameters (volatility,

drift, jump measure, etc.) of the forward process (stock prices/particle fluctuations/dispersal of animals/population dynamics, etc.);

- 2) Reflected BSDEs on general filtered spaces (see [13,15,18,20,29]) and nonlinear stopping problems - the strategy of the player is to stop the game in an optimal moment (stopping time) to maximize the pay-off function, which is described via nonlinear expectation operator (see [12]);
- 3) BSDEs with two reflecting barriers on general filtered spaces (see [3,10,14,18,19,23]) and nonlinear zero-sum Dynkin games (see [10,14,19]). In this optimization problem we have two players with strategies consisting of stopping times. When player 1 stops the game at time  $\tau$  before player 2, then player 1 gains reward  $A(\tau)$  described via nonlinear expectation of a stochastic process, and player 2 loses this amount (zero-sum game). Otherwise, player 1 gains  $B(\tau)$  and player 2 loses  $B(\tau)$ ;
- 4) Nonzero-sum Dynkin games problem (see [16]). It is the game similar to the one described in 3), but with the considerable change: when player 1 gains  $A(\tau)$ , then player 2 does not necessarily lose  $A(\tau)$ , and when player 1 gains  $B(\tau)$ , then player 2 does not necessarily lose  $B(\tau)$ . An open problem is the form of the associated BSDE;
- 5) Second order reflected BSDEs (see [24]) and robust optimal stopping problems and robust Dynkin games (see [1]). In these problems, the strategies of the player/players consist of stopping times and family of probability measures (uncertainty) describing the gain functional;
- 6) Quasi-linear reflected BSDEs and impulse control problems (see [7]), where the strategy of the player consists of an increasing sequence  $(\tau_n)$  of stopping times (intervention times) and a sequence  $(c_n)$  of appropriately measurable random variables (cost of the intervention). At any time  $\tau_n$  the player may shift the process with some cost  $c_n$  (so called control action);
- 7) Multidimensional reflected BSDEs (see [2]) in random convex and non-convex closed sets and the switching problem (see [17]), where the strategy of the player is again an increasing sequence  $(\tau_n)$  of stopping times. In this case at any time  $\tau_n$  the player switches (with some cost  $c_n$ ) to the different cost/profit process (generator).

### 1.3. Work plan

See Sections 1.1, 1.2.

## 1.4. Literature

1. Bayraktar, E., Yao, S.: On the robust Dynkin game. *Ann. Appl. Probab.* 27 (2017), no. 3, 1702-1755.
2. Bensoussan, A., Li, Y., Yam, S.C.P.: Backward stochastic dynamics with a subdifferential operator and non-local parabolic variational inequalities. *Stochastic Process. Appl.* 128 (2018), no. 2, 644-688.
3. Buckdahn, R., Li, J., Probabilistic interpretation for systems of Isaacs equations with two reflecting barriers. *NoDEA Nonlinear Differential Equations Appl.* 16 (2009), no. 3, 381-420
4. Buckdahn, R., Li, J., Peng, S., Rainer, C.: Mean-field stochastic differential equations and associated PDEs. *Ann. Probab.* 45 (2017), no. 2, 824-878.
5. Cvitanic J., Karatzas, I., Backward Stochastic Differential Equations with Reflection and Dynkin Games. *Ann. Probab.* 19 (1996) 2024-2056.
6. Cvitanic, J., Karatzas, I., Soner, H. M., Backward stochastic differential equations with constraints on the gains-process. *Ann. Probab.* 26 (1998), no. 4, 1522-1551.
7. Djehiche, B., Hamadene, S., Hdhiri, I.: Stochastic impulse control of non-Markovian processes. *Appl. Math. Optim.* 61 (2010), no. 1, 1-26.
8. Duffie, D., Epstein, L., Stochastic Differential Utility, *Econometrica* 60 (1992) 353-394.
9. Duffie, D., Epstein, L., Asset Pricing with Stochastic Differential Utility. *Rev. Financial Stud.* 5 (1992) 411-436.
10. Dumitrescu, R., Quenez, M.C., Sulem, A., Generalized Dynkin games and doubly reflected BSDEs with jumps. *Electron. J. Probab.* 21 (2016) 1-32.
11. El Karoui, N., Kapoudjian, C., Pardoux, E., Peng, S., Quenez, M.-C.: Reflected solutions of backward SDEs, and related obstacle problems for PDE's. *Ann. Probab.* 25 (1997) 702-737.
12. El Karoui, N., Quenez, M.-C., Non-linear pricing theory and backward stochastic differential equations. *Lecture Notes in Math.* 1656 (1997) 191-246.
13. Grigorova, M., Imkeller, P., Offen, E., Ouknine, Y., Quenez, M.-C.: Reflected BSDEs when the obstacle is not right-continuous and optimal stopping. *Ann. Appl. Probab.* 27 (2017), no. 5, 3153-3188
14. Hamadene, S., Mixed zero-sum stochastic differential game and American game options. *SIAM J. Control Optim.* 45 (2006), no. 2, 496-518.
15. Hamadene, S., Lepeltier, J.-P.: Reflected BSDEs and mixed game problem. *Stochastic Process. Appl.* 85 (2000), no. 2, 177-188.
16. Hamadène, S., Mohammed, H.: The multiplayer nonzero-sum Dynkin game in continuous time. *SIAM J. Control Optim.* 52 (2014), no. 2, 821-835.
17. Hamadene, S., Zhang, J., Switching problem and related system of reflected backward SDEs. *Stochastic Process. Appl.* 120 (2010), no. 4, 403-426.
18. Klimsiak, T., Reflected BSDEs on filtered probability spaces. *Stochastic Process. Appl.* 125 (2015) 4204-4241.
19. Klimsiak, T.: Non-semimartingale solutions of reflected BSDEs and applications to Dynkin

- games. *Stochastic Process. Appl.* 134 (2021) 208–239
20. Karatzas, I., Zamfirescu, I.-M.: Martingale approach to stochastic differential games of control and stopping. *Ann. Probab.* 36 (2008), no. 4, 1495–1527.
  21. Kramkov, D., Pulido, S.: A system of quadratic BSDEs arising in a price impact model. *Ann. Appl. Probab.* 26 (2016), no. 2, 794–817.
  22. Liang, G., Lyons, T., Qian, Z., Backward stochastic dynamics on a filtered probability space. *Ann. Probab.* 39 (2011) 1422–1448.
  23. Lin, Q., Nash Equilibrium Payoffs for Stochastic Differential Games with two Reflecting Barriers. *Advances in Applied Probability* 47 (2015) 355–377.
  24. Matoussi, A., Possamai, D., Zhou, C.: Second order reflected backward stochastic differential equations. *Ann. Appl. Probab.* 23 (2013), no. 6, 2420–2457
  25. Pardoux, E., Peng, S., Adapted solution of a backward stochastic differential equation. *Systems Control Lett.* 14 (1990) 55–61.
  26. Pardoux, E., Peng, S., Backward SDE's and quasilinear PDE's. *Lecture Notes in Control and Inform. Sci.* 176 (1992) 200–176.
  27. Pardoux, E., Zhang, S., Generalized BSDEs and Nonlinear Neumann Boundary Value Problems. *Probab. Theory Relat. Fields* 110 (1998) 535–558
  28. Peng, S., Probabilistic interpretation for systems of quasilinear parabolic partial differential equations. *Stochastics Stochastics Rep.* 37 (1991) 61–74.
  29. Ren, Y., Xia, N., Generalized reflected BSDE and an obstacle problem for PDEs with a nonlinear Neumann boundary condition. *Stoch. Anal. Appl.* 24 (2006), no. 5, 1013–1033.
  30. Rozkosz, A., Backward SDEs and Cauchy problem for semilinear equations in divergence form. *Probab. Theory Relat. Fields* 125 (2003) 393–401.
  31. Soner, H. M., Touzi, N., Zhang, J. Wellposedness of second order backward SDEs. *Probab. Theory Related Fields* 153 (2012), no. 1–2, 149–190.

**1.5. Required initial knowledge and skills of the PhD candidate:** analytic thinking, willingness of self-study, understanding of mathematical analysis, solid knowledge on probability theory, functional analysis, and basic knowledge on stochastic processes, differential equations.

**1.6. Expected development of the PhD candidate's knowledge and skills:** conducting research on a high level, acquiring skills to present scientific achievements on professional level, acquiring advanced knowledge on stochastic equations, stochastic processes, optimization problems and stochastic analysis.